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COMMENT

A diagrammatic representation of relations between critical exponents

I G Enting

Research School of Physical Sciences, The Australian National University, Canberra, ACT 2600 Australia

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Abstract. It is shown that if exponent estimates are represented by lines on a graph then a particularly simple representation of exponent inequalities and scaling relations results. The exponents α', β, γ', δ are considered and the three-state Potts model is used to illustrate the technique.

When series expansions for lattice models in statistical mechanics are analysed, the critical exponent estimates that are obtained each have an associated range of uncertainty. As an example we consider the estimates obtained by Straley and Fisher (1973) for the three-state Potts model $\alpha' = 0{\cdot}05 \pm 0{\cdot}10$, $\beta = 0{\cdot}10 \pm 0{\cdot}01$, $\gamma' = 1{\cdot}5 \pm 0{\cdot}2$ and the estimate $\delta = 15{\cdot}0 \pm 0{\cdot}5$ obtained by Enting (1974a). These exponents are however subject to the constraints:

$$\alpha' \geqslant 2 - 2\beta - \gamma' \qquad \text{(Rushbrooke 1963)} \qquad (1)$$

$$\alpha' \geqslant 2 - (1+\delta)\beta \qquad \text{(Griffiths 1965)} \qquad (2)$$

and scaling theory predicts that the equalities in (1) and (2) hold. Enting (1974b) compared the estimates above against the constraints and suggested that Straley and Fisher had most probably underestimated α'. This conclusion was reached by fitting possible values from the exponents and finding which set of three exponents gave the most consistent set. The purpose of this comment is to describe a graphical technique which simplifies this type of fit and which, more importantly, gives a way of communicating the fitting process.

Each exponent estimate is represented on the graph by a straight line so that one has

$$\alpha' \text{ line:} \qquad y = \alpha' \qquad (3a)$$

$$\beta \text{ line:} \qquad x = \beta \qquad (3b)$$

$$\gamma' \text{ line:} \qquad 2x + y = 2 - \gamma' \qquad (3c)$$

$$\delta \text{ line:} \qquad x(1+\delta) + y = 2. \qquad (3d)$$

In terms of these lines, the scaling prediction is that all four lines will pass through a common point as shown in figure 1(a). For the Rushbrooke or Griffiths inequalities to be satisfied the α', β, γ' or α', β, δ lines must intersect as shown in figure 1(b) (with equality when the enclosed triangle vanishes). Violation of the Rushbrooke or

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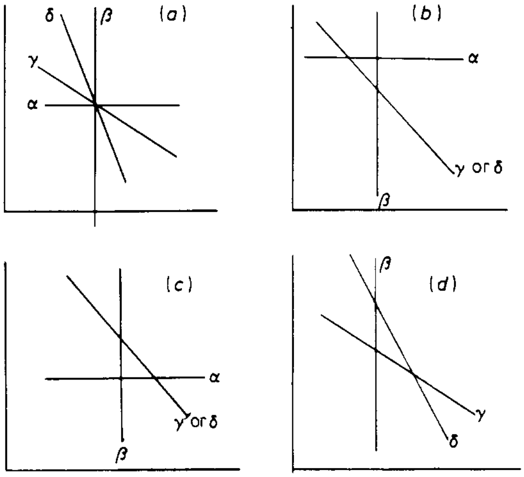

Figure 1. Relationships between exponent lines. (*a*) If scaling holds then all four lines pass through one point. (*b*) If the Rushbrooke inequality (or Griffiths inequality) is satisfied then the α' line passes above or through the intersection of the β line with the γ' line (or δ line). (*c*) If the Rushbrooke (or Griffiths) inequality is violated the α' line passes below the intersection of the β line and γ' (or δ) line. (*d*) If inequality (4) is satisfied then the γ' line passes below (or through) the intersection of β and δ lines.

Griffiths inequalities corresponds to intersections of the form shown in figure 1(*c*). Griffiths (1965) has also derived the inequality:

$$\gamma' \geqslant \beta(\delta - 1). \tag{4}$$

This inequality is satisfied if the γ', β, δ lines intersect as shown in figure 1(*d*). The conditions under which inequalities (1), (2) and (4) can be proved are listed by Griffiths (1972).

Once individual exponent estimates are represented by lines on a graph it is obvious that a range of estimates is simply represented by a band. The exponent estimates quoted above are shown on figure 2. The figure shows that the departures from scaling (and violations of (1) and (2)) seem to be due to the estimates of α' being too small. This is even more obvious if one uses the initial estimate $\gamma' = 1{\cdot}5 \pm 0{\cdot}1$ obtained by Straley and Fisher (1973).

The mathematical basis of these diagrams is the fact that the inequalities (1), (2) and (4) reduce to equalities if scaling holds and that scaling theory describes the critical exponents α', β, γ', δ in terms of two parameters a, b (see for example Hankey and Stanley 1972).

$$\alpha' = (2a - 1)/a \tag{5a}$$

$$\beta = (1 - b)/a \tag{5b}$$

$$\gamma' = (2b - 1)/a \tag{5c}$$

$$\delta = b/(1 - b). \tag{5d}$$

Any exponent estimate will correspond to a one-dimensional curve of possible values of (a, b) in the two-dimensional a–b plane. The values of a, b are determined by the intersections of two such curves. Once the uncertainties are included the allowed

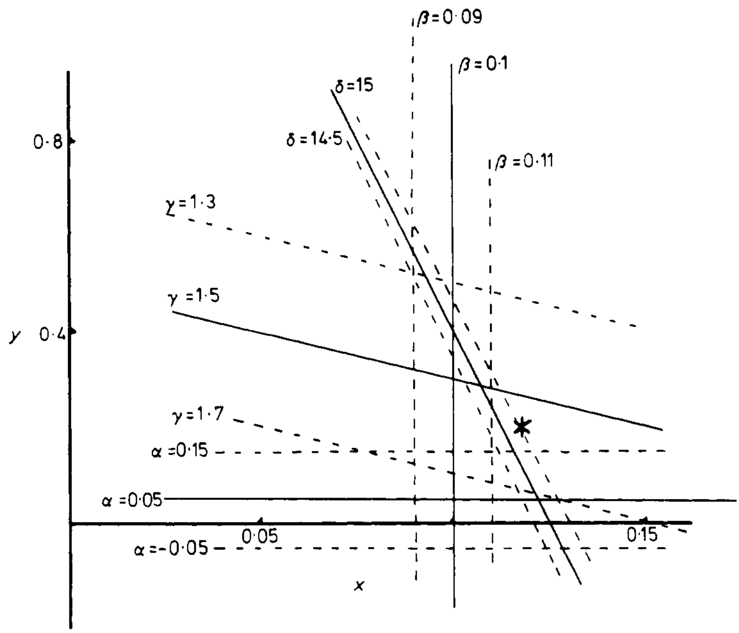

Figure 2. The exponent estimates for the three-state Potts model. The point * corresponds to the renormalisation group calculations of Berker and Wortis.

values of a, b are those within the intersections of 'bands' in the a–b plane. Comparison of the areas of overlap of various pairs of curves enables us to make statements about which exponent estimates give the tightest constraints on other exponents. In figure 2 it is apparent that β and δ are the exponents for which the estimates most tightly define a and b.

The variables which are actually used to display the curves can be chosen according to the requirements of the problem in hand. The variables x, y of figure 2 were chosen so that the α', β, γ' and δ curves were all straight lines:

$$x = (1-b)/a \tag{6a}$$

$$y = (2a-1)/a. \tag{6b}$$

When renormalisation group techniques are used to obtain critical exponents, scaling is built into the calculation and so one can regard the renormalisation group as giving a single pair (a, b) which would be represented by a point on the graphs. As an example figure 2 shows the results of calculations by Berker and Wortis (1976). The exponents β, α' are given by the coordinates of the point. The exponent γ' is obtained by constructing a line of gradient -2 through the point. From (3c), the x intercept is $2-\gamma'$. The exponent δ is obtained by constructing a line through $(0, 2)$ and the renormalisation group point. The gradient is $-(1+\delta)$.

It should be noted there have been a large number of other estimates of Potts model exponents, most of which gave values of α' exceeding the values estimated by Straley and Fisher. In particular de Neef and Enting estimated $\alpha' = 0{\cdot}42 \pm 0{\cdot}05$ which will be seen to be consistent with the estimates of β, γ', δ plotted on figure 2.

Once the diagrammatic technique is related to the two-parameter scaling formalism it is obvious that the technique can be extended into other situations. Even in three-parameter scaling (e.g. tricritical scaling) where each exponent estimate would correspond to a surface in a three-dimensional space, suitable cross sections may be useful for displaying the relationships between exponent estimates.

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